

The universal n -pointed surface bundle only has n sections

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Abstract

The classifying space $\mathrm{BDiff}(S_{g,n})$ of the orientation-preserving diffeomorphism group of the surface $S_{g,n}$ of genus $g > 1$ with n ordered marked points has a universal bundle

$$S_g \rightarrow \mathrm{UDiff}(S_{g,n}) \xrightarrow{\pi} \mathrm{BDiff}(S_{g,n}).$$

The fixed n points provide n sections s_i of π . In this paper we prove a conjecture of R. Hain that any section of π is homotopic to some s_i . Let $\mathrm{PConf}_n(S_g)$ be the ordered n -tuples of distinct points on S_g . As part of the proof of Hain's conjecture, we prove a result of independent interest: any surjective homomorphism $\pi_1(\mathrm{PConf}_n(S_g)) \rightarrow \pi_1(S_g)$ is equal to one of the forgetful homomorphisms $\{p_i : \pi_1(\mathrm{PConf}_n(S_g)) \rightarrow \pi_1(S_g)\}$, possibly post-composed with an automorphism of $\pi_1(S_g)$, $1 \leq i \leq n$. Using similar arguments, we then show that the universal surface bundle that fixes n points as a set does not have any section.

1 Introduction

Let $\mathrm{Diff}(S_{g,n})$ denote the orientation-preserving diffeomorphism group of the surface S_g of genus $g > 1$ that fixes n distinct points $\{x_1, x_2, \dots, x_n\} \subset S_g$ individually. There is a fiber bundle

$$S_g \rightarrow \mathrm{UDiff}(S_{g,n}) \xrightarrow{\pi} \mathrm{BDiff}(S_{g,n}) \quad (1)$$

which is universal in the sense that any S_g -bundle endowed with n disjoint sections is a pullback of this bundle. Since $\mathrm{Diff}(S_{g,n})$ fixes the n points x_1, x_2, \dots, x_n , we associate n points on each fiber, i.e. n disjoint sections which will be called s_1, s_2, \dots, s_n . A natural question is: are there more sections?

R. Hain conjectured that every section of (1) is homotopic to one of these n sections. This is the main theorem of this paper.

Theorem 1.1 (Classification of sections). *For any $n \geq 0$ and $g > 2$, every section of the universal bundle (1) is homotopic to s_i for some $i \in \{1, 2, \dots, n\}$. For $g = 2$, there are precisely $2n$ homotopy classes of sections of the universal bundle (1).*

We also know that each sections s_i has nontrivial self-intersection from cohomology computation, we have the following corollary.

Corollary 1.2. *The universal bundle (1) does not admit $n + 1$ disjoint sections.*

What if we only fix the n points as a set? More precisely, let $\text{Diff}(S_{g,\bar{n}})$ denote the orientation-preserving diffeomorphism group of the surface S_g of genus $g > 1$ that fixes n points $\{x_1, x_2, \dots, x_n\} \subset S_g$ as a set. There is a fiber bundle

$$S_g \rightarrow \text{UDiff}(S_{g,\bar{n}}) \rightarrow \text{BDiff}(S_{g,\bar{n}}). \quad (2)$$

We also have the following result.

Theorem 1.3 (No Sections Theorem). *For any $n > 1$ and $g > 1$, there is no section of the bundle (2).*

We will see below that Hain's conjecture can be represented both in terms of mapping class groups and also in terms of moduli spaces. Let $\mathcal{M}_{g,m,n}$ be the moduli space of smooth genus g Riemann surfaces with $m + n$ distinct points, m labelled and n unlabelled. Earle-Kra [EK76] proved that the only holomorphic section of the forgetful map $f : \mathcal{M}_{g,m,n} \rightarrow \mathcal{M}_{g,m,0}$ occurs when $g = 2$ and $n = 6$. This section is constructed by marking all six Weierstrass points.

Corollary 1.2 and Theorem 1.3 give a topological proof of the fact that there is no continuous section of $\mathcal{M}_{g,m+1,0} \rightarrow \mathcal{M}_{g,m,0}$ and there is no continuous section of $\mathcal{M}_{g,0,n} \rightarrow \mathcal{M}_{g,0,1}$, respectively.

The key ingredient is the following question. Let $\text{PConf}_n(S_g)$ be the set of ordered n -tuples of distinct points on S_g .

Question 1.4. *How many homotopy classes of maps from $\text{PConf}_n(S_g)$ to S_g can we have?*

There are n forgetful maps $p_i : \text{PConf}_n(S_g) \rightarrow S_g$. Do we have more maps?

Remark. The following graph is a cartoon version of what the following theorem talks about. Any homomorphism from pure braid with n strings to pure braid with 1 string is either having \mathbb{Z} image or a composition of forgetting homomorphism f_i and a homomorphism $\pi_1(S_g) \rightarrow \pi_1(S_g)$.

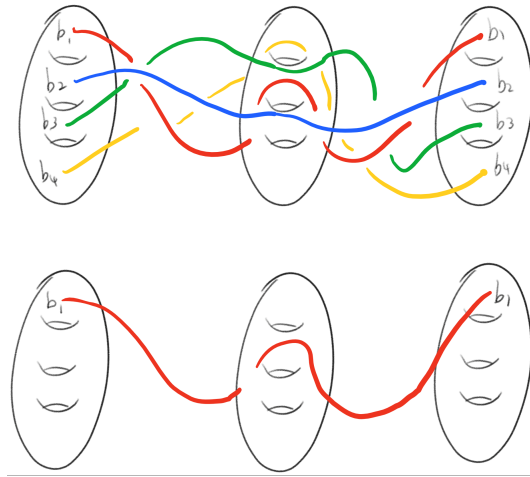


Figure 1: Braid group section homomorphism

We answer Question 1.4 by the following classification theorem.

Theorem 1.5 (Classification of homomorphisms $\pi_1(\text{PConf}_n(S_g)) \rightarrow \pi_1(S_g)$). *Let $g > 1$ and $n > 0$. Let $R : \pi_1(\text{PConf}_n(S_g)) \rightarrow \pi_1(S_g)$ be a homomorphism. The following hold:*

1) If R is surjective, then R is one of the forgetful homomorphisms $\pi_1(P\text{Conf}_n(S_g)) \xrightarrow{p_i} \pi_1(S_g)$ up to post-composed with an automorphism A of $\pi_1(S_g)$.

2) If $\text{Image}(R)$ is not a cyclic group, the homomorphism $\pi_1(P\text{Conf}_n(S_g)) \rightarrow \pi_1(S_g)$ factors through one of the forgetful homomorphisms p_i .

Organization of the paper

In Section 2, we translate the problem to group theory. The cohomological structure of configuration space is computed in Section 3 in order to study the possible homomorphisms between $\pi_1(P\text{Conf}_n(S_g))$ and $\pi_1(S_g)$, and prove Theorem 1.5. The method that is used in proving Theorem 1.5 is similar to the one used in [Joh99]. In Section 4, we deal with different homomorphisms R as discussed in Theorem 1.5 to prove Theorem 2.2 and in Section 5 we prove 2.3. In the end, we talk about the meaning of conjugation and prove Theorem 1.1 and 1.3 in Section 6.

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2 Translating the problem to group theory

2.1 Translation the general question

Let $\text{Diff}(S_g)$ denote the orientation-preserving diffeomorphism group of the surface S_g of genus $g > 1$. We have the universal $\text{Diff}(S_g)$ principal bundle

$$\text{Diff}(S_g) \rightarrow \text{EDiff}(S_g) \rightarrow \text{BDiff}(S_g)$$

Here $\text{EDiff}(S_g)$ is the total space of the universal $\text{Diff}(S_g)$ bundle, i.e. a contractible principal $\text{Diff}(S_g)$ bundle. Let $\text{UDiff}(S_g) = \text{EDiff}(S_g) \times_{\text{Diff}(S_g)} S_g$ be the universal surface bundle.

$$S_g \rightarrow \text{UDiff}(S_g) \xrightarrow{\zeta} \text{Diff}(S_g)$$

$\text{BDiff}(S_g)$ classifies surface bundles, which means any surface bundle $S_g \rightarrow E \rightarrow B$ is the pullback of ζ via a continuous map $f : B \rightarrow \text{BDiff}(S_g)$.

By a theorem of Earle and Eells [EE67] stating that $\text{Diff}_0(S_g)$, i.e. the identity component of $\text{Diff}(S_g)$, is contractible, we have $\text{BDiff}(S_g) = K(\text{Mod}_g, 1)$. Therefore any map $f : B \rightarrow \text{BDiff}(S_g)$ is determined by the monodromy representation

$$f : \pi_1(M) \rightarrow \text{Mod}_g.$$

We have the following correspondence:

$$\left\{ \begin{array}{c} \text{Conjugacy classes of} \\ \text{representation} \\ f : \pi_1(B) \rightarrow \text{Mod}_g \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{oriented } S_g\text{-bundles over} \\ B \end{array} \right\} \quad (3)$$

Let $\text{Diff}(S_{g,1})$ be the orientation-preserving diffeomorphism group of the surface S_g of genus $g > 1$ fixing one point. There is a natural inclusion $\text{Diff}(S_{g,1}) \hookrightarrow \text{Diff}(S_g)$.

Proposition 2.1.

$$\text{UDiff}(S_g) = \text{BDiff}(S_{g,1})$$

Proof.

$$\begin{aligned} \text{UDiff}(S_g) &= \text{EDiff}(S_g) \times_{\text{Diff}(S_g)} S_g && \text{By definition} \\ &= \text{EDiff}(S_g) \times_{\text{Diff}(S_g)} \text{Diff}(S_g)/\text{Diff}(S_{g,1}) && \text{Because } S_g = \text{Diff}(S_g)/\text{Diff}(S_{g,1}) \\ &\cong \text{EDiff}(S_g)/\text{Diff}(S_{g,1}) && \text{Diff}(S_{g,1}) \text{ is a subgroup of } \text{Diff}(S_g) \\ &\cong \text{BDiff}(S_{g,1}) && \text{EDiff}(S_g) \text{ is contractible} \end{aligned}$$

□

Proposition (2.1) implies that the universal surface bundle is actually $K(\text{Mod}_{g,1}, 1)$. The group theoretic universal bundle is the following exact sequence:

$$1 \rightarrow \pi_1(S_g) \rightarrow \text{Mod}_{g,1} \rightarrow \text{Mod}_g \rightarrow 1 \quad (4)$$

General main problem. For any surface bundle with base B , we denote $G = \pi_1(B)$ and the following diagram is a pullback from (4) of the monodromy representation of $G \rightarrow \text{Mod}_g$.

$$1 \rightarrow \pi_1(S_g) \rightarrow \Gamma_G \rightarrow G \rightarrow 1.$$

How many splittings can we find for this exact sequence?

For $G = \text{Mod}_g$, it is well-known that this exact sequence has no splitting. This is the $n = 0$ case of Theorem 1.1. This result can be found e.g. in [FM12] Corollary 5.11.

In [Mes90], G. Mess claimed the non splitting theorem when G is Torelli group but there exists counter examples for the proof. In [Che17], we prove the case when G is finite index subgroup of Mod_g and $g > 2$.

By the above correspondence (3), any extension Γ_G is a pullback of the universal bundle. By the property of the pullback diagram, finding a section of π is the same as finding a homomorphism p that satisfies $\pi_1 \circ p = m$ in the following diagram.

$$\begin{array}{ccc} \Gamma_G & \longrightarrow & \text{Mod}_{g,1} \\ \downarrow \pi & \nearrow p & \downarrow \pi_1 \\ G & \xrightarrow{m} & \text{Mod}_g \end{array}$$

2.2 Translation our question

Let $\text{Mod}_{g,n}$ denote the mapping class group of $S_{g,n}$, which is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of S_g that fixes n points $\{x_1, x_2, \dots, x_n\} \subset S_g$ as a set.

$$\text{Mod}_{g,n} = \text{Diff}(S_{g,n})/\text{Diff}_0(S_{g,n})$$

Here $\text{Diff}_0(S_{g,n})$ is the subgroup of $\text{Diff}(S_{g,n})$ consisting of elements that are isotopic to the identity.

Let $\text{PMod}_{g,n}$ be the pure mapping class group of $S_{g,n}$, which is defined to be the subgroup of $\text{Mod}_{g,n}$ consisting of elements that fix $\{x_i\}$ individually. Thus $\text{Mod}_g = \text{PMod}_{g,0} = \text{BMod}_{g,0}$. There are two natural forgetful homomorphisms $\pi'_n : \text{Mod}_{g,n} \rightarrow \text{Mod}_g$ and $\pi_n : \text{PMod}_{g,n} \rightarrow \text{Mod}_g$. Those homomorphisms are realized by forgetting the fixed points.

In this paper, we study the splitting problem in the cases that $G = \text{PMod}_{g,n}$ and $G = \text{Mod}_{g,n}$. Since $\text{Diff}_0(S_{g,n})$ and $\text{Diff}_0(S_{g,\bar{n}})$ are also contractible, we have that $\text{BDiff}(S_{g,n}) = K(\text{PMod}_{g,n}, 1)$ and $\text{BDiff}(S_{g,\bar{n}}) = K(\text{Mod}_{g,\bar{n}}, 1)$. The monodromy representations $\pi'_n : \text{PMod}_{g,n} \rightarrow \text{Mod}_g$ and $\pi'_n : \text{Mod}_{g,n} \rightarrow \text{Mod}_g$ give us two S_g bundles (1) and (2). Let $\text{CPMod}_{g,n} = \pi_1(\text{UDiff}(S_{g,n}))$ and $\text{CMod}_{g,n} = \pi_1(\text{UDiff}(S_{g,\bar{n}}))$. The long exact sequences of the two bundles are shortened to the following due to the fact that all the spaces involved are $K(\pi, 1)$ -spaces.

$$1 \rightarrow \pi_1(S_g) \rightarrow \text{CPMod}_{g,n} \xrightarrow{u} \text{PMod}_{g,n} \rightarrow 1. \quad (5)$$

and

$$1 \rightarrow \pi_1(S_g) \rightarrow \text{CMod}_{g,n} \xrightarrow{u'} \text{Mod}_{g,n} \rightarrow 1. \quad (6)$$

Let $\text{PConf}_n(S_g)$ be the ordered n -tuples of distinct points on S_g . There is a natural permutation group Σ_n free action on $\text{PConf}_n(S_g)$. Let $\text{Conf}_n(S_g) = \text{PConf}_n(S_g)/\Sigma_n$ be the unordered n -tuples of distinct points on S_g . We will denote $PB_n(S_g) := \pi_1(\text{BConf}_n(S_g))$ and $B_n(S_g) := \pi_1(\text{Conf}_n(S_g))$ similar to the notation of pure braid group and braid group.

We have the following exact sequence describing the kernel of the forgetful homomorphism π_n and π'_n .

$$\begin{aligned} 1 \rightarrow PB_n &\xrightarrow{\text{point pushing}} \text{PMod}_{g,n} \xrightarrow{\pi_n} \text{Mod}_g \rightarrow 1 \\ 1 \rightarrow B_n &\xrightarrow{\text{point pushing}} \text{Mod}_{g,n} \xrightarrow{\pi'_n} \text{Mod}_g \rightarrow 1 \end{aligned}$$

Under the translate at the end of the previous subsection, the problem of finding sections of (1) and (2) is the same as finding homomorphisms p and p' that make the following diagrams (7) and (8) commute. We denote R and R' to be the restriction of p and p' on the kernels $PB_n(S_g)$ and $B_n(S_g)$.

$$\begin{array}{ccccc} 1 \rightarrow PB_n(S_g) & \longrightarrow & \text{PMod}_{g,n} & \xrightarrow{\pi_n} & \text{Mod}_g \longrightarrow 1 \\ \downarrow \wr_R & & \downarrow \wr_p & & \downarrow = \\ 1 \rightarrow \pi_1(S_g) & \longrightarrow & \text{Mod}_{g,1} & \xrightarrow{\pi_1} & \text{Mod}_g \longrightarrow 1 \end{array} \quad (7)$$

$$\begin{array}{ccccc} 1 \rightarrow B_n(S_g) & \longrightarrow & \text{Mod}_{g,n} & \xrightarrow{\pi'_n} & \text{Mod}_g \longrightarrow 1 \\ \downarrow \wr_{R'} & & \downarrow \wr_{p'} & & \downarrow = \\ 1 \rightarrow \pi_1(S_g) & \longrightarrow & \text{Mod}_{g,1} & \xrightarrow{\pi_1} & \text{Mod}_g \longrightarrow 1 \end{array} \quad (8)$$

Let $\text{PMod}_{g,n} \xrightarrow{p_i} \text{Mod}_{g,1}$ be the forgetful homomorphism that forgets the fixing points $\{x_1, \dots, x_i, \dots, x_n\}$.

Theorem 1.1 is thus equivalent to the following.

Theorem 2.2. *When $g > 2$ and $n \geq 0$, every homomorphism p that satisfies the diagram (7) is conjugate by an element A in $\pi_1(S_g)$ to one of the forgetful homomorphism p_i .*

Theorem 1.3 is thus equivalent to the following.

Theorem 2.3. *When $g > 1$ and $n > 1$, there is no homomorphism p' that satisfies the diagram (8).*

After analyzing the possibilities for R , we later want to extend R to p . We then rule out most cases because the homomorphism R have to be equivariant with respect to the action of Mod_g .

3 Classification of homomorphisms $\pi_1(\text{PConf}_n(S_g)) \xrightarrow{R} \pi_1(S_g)$

This section is divided into 3 parts. We first do a cohomology computation of $\text{PConf}_n(S_g)$, we then study a property of the cohomology, and finally we combine them to prove Theorem 1.5. The reason to do the computation is to use an argument of [Joh99] to show that there is some cohomological constraint on the homomorphism $PB_n(S_g) \xrightarrow{R} \pi_1(S_g)$.

Throughout the paper, all cohomology are computed with \mathbb{Q} coefficients. We also assume throughout that $g > 1$ and $n > 0$.

3.1 Step 1: First cohomology of $\text{PConf}_n(S_g)$

In this subsection, for a topological space X , we identify $H^1(X; \mathbb{Q}) \cong \text{Hom}(H_1(X), \mathbb{Q})$.

Let $\{x_1, \dots, x_n\} \subset S_g$ be n disjoint points on S_g . Let $S_{g,n} = S_g - \{x_1, \dots, x_n\}$ be the n punctured genus g surface, we have the following fibration:

$$S_{g,n} \rightarrow \text{PConf}_{n+1}(S_g) \xrightarrow{F} \text{PConf}_n(S_g) \quad (9)$$

Lemma 3.1. *There is a natural embedding of $H^1(S_g) \hookrightarrow H^1(S_{g,n})^{PB_n(S_g)}$. We have the following isomorphism.*

$$H^1(S_{g,n})^{PB_n(S_g)} \cong H^1(S_g)$$

Proof. Let $(x_1, \dots, x_n) \in \text{PConf}_n(S_g)$ be our base point for computing fundamental group. $PB_n(S_g)$ is generated by simple closed curves based at $\{x_1, \dots, x_n\}$. For any loop b at some x_i , we have an element $\text{Push}(b)$ in $\text{PMod}_{g,n}$. The way we think of $\text{Push}(b)$ informally is that we place our finger on x_i and push x_i along b , dragging the rest of the surface along as we go. If we forget the basepoint x_i , $\text{Push}(b)$ is isotopic to the identity. Therefore the action of $PB_n(S_g)$ on $H_1(S_g)$ is trivial.

The following figure provides the necessary notation we need for the proof. Let c_1, c_2, \dots, c_n be the n loops around x_1, \dots, x_n .

Let b be as in the picture, representing a nontrivial simple closed curve at x_1 and we have that (for example see [FM12] Fact 4.7)

$$\text{Push}(b) = T_{b_2}^{-1} T_{b_1}.$$

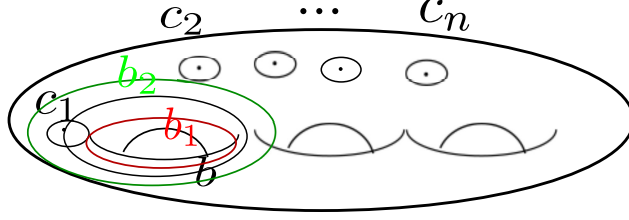


Figure 2: point push

Therefore for any homology class $c \in H_1(S_{g,n})$, denoting by $\langle b, c \rangle$ the algebraic intersection number.

$$\text{Push}(b)(c) = \langle b, c \rangle c_1 + c$$

All the other generators in $PB_n(S_g)$ act similarly. $f \in H^1(S_{g,n})^{PB_n(S_g)}$ implies $f(c) = f(g(c))$ for $g \in PB_n(S_g)$, therefore $f(c) = f(\langle b, c \rangle c_1 + c)$. We can find b and c such that $\langle b, c \rangle = 1$, so we should have $f(c_1) = 0$. By the same argument, we have $f(c_i) = 0$. Also we know that the image of $H^1(S_g) \rightarrow H^1(S_{g,n})$ is invariant, whose elements are those functionals that take 0 value on all c_i . Therefore we get the result. \square

Under the action of $PB_n(S_g)$ on $H^1(S_{g,n})$, the subspace $H^1(S_g) \subset H^1(S_{g,n})$ is an invariant subspace. Therefore $PB_n(S_g)$ acts naturally on the quotient group $H^1(S_{g,n})/H^1(S_g)$.

Lemma 3.2. *The action of $PB_n(S_g)$ on $H^1(S_{g,n})/H^1(S_g)$ is trivial.*

Proof. All we have to do is to show that for any $f \in H^1(S_{g,n})$ and $g \in PB_n(S_g)$, we have $f - f \circ g \in H^1(S_g)$.

Since the image of $H^1(S_g) \rightarrow H^1(S_{g,n})$ is all $f \in \text{Hom}(H_1(S_{g,n}), \mathbb{Q})$ that take zero value on c_i , i.e. the small loops around x_i , we need to prove that $(f - f \circ g)(c_i) = 0$ or equivalently $f(c_i) = f \circ g(c_i)$ for any c_i . However we know that the action of $PB_n(S_g)$ does not change the tiny loop c_i , so $g(c_i) = c_i \in H_1(S_g)$. \square

Lemma 3.3.

$$H^1(\text{PConf}_n(S_g)) \cong \bigoplus_{i=1}^n H^1(S_g)$$

Later on, H_i^1 will denote the i th copy of the subspace. Also we have $H^2(\text{PConf}_n(S_g)) \rightarrow H^2(\text{PConf}_{n+1}(S_g))$ is injective.

Proof. Applying the Serre spectral sequence to the fiber bundle (9), the filtration of the first cohomology is:

$$0 \rightarrow H^1(\text{PConf}_n(S_g)) \rightarrow H^1(\text{PConf}_{n+1}(S_g)) \xrightarrow{e} H^1(S_{g,n})^{PB_n(S_g)} \quad (10)$$

$H^1(S_{g,n})^{PB_n(S_g)} \cong H^1(S_g)$ by Lemma 3.1. The arrow e is surjective because the projection

$$\text{PConf}_{n+1}(S_g) \xrightarrow{p_{n+1}} S_g$$

gives us a copy of $H^1(S_g)$ that maps to the whole group $H^1(S_{g,n})^{PB_n(S_g)}$. Therefore, this exact sequence is also splitting. We have the following short exact sequence.

$$0 \rightarrow H^1(\text{PConf}_n(S_g)) \rightarrow H^1(\text{PConf}_{n+1}(S_g)) \xrightarrow{e} H^1(S_g) \rightarrow 0$$

Using induction, we can get the result. Notice that since the arrow e in exact sequence 10 is surjective, the differential $H^1(S_{g,n})^{PB_n(S_g)} \xrightarrow{d_2} H^2(\text{PConf}_n(S_g))$ is 0. Therefore $H^2(\text{PConf}_n(S_g)) \rightarrow H^2(\text{PConf}_{n+1}(S_g))$ is injective. \square

3.2 Step 2: Second cohomology of $\text{PConf}_n(S_g)$

Now we move on to the discussion of $H^2(\text{PConf}_n(S_g))$. Instead of determining them all, we focus on the image of the cup product from the first cohomology. The image has the following potential image:

$$\mathbb{Q}^n \oplus \bigoplus_{i \neq j} H_i^1 \otimes H_j^1 \xrightarrow{C_n} H^2(\text{PConf}_n(S_g))$$

Here $\mathbb{Q}^n = \bigoplus_{i=1}^n \mathbb{Q}p_i^*[S_g]$, denoting the span of the n pullbacks of the projection $\text{PConf}_n(S_g) \xrightarrow{p_i} S_g$. The homomorphism C_n on $H_i^1 \otimes H_j^1$ is the cup product of every two copies of $H^1(S_g)$. Since the image of $\wedge^2 H_i^1$ in $H^2(\text{PConf}_n(S_g))$ is a multiple of $p_i^*[S_g]$, we know that:

Remark. The $\text{Im}(C_n)$ is equal to the image of $\wedge^2 H^1(\text{PConf}_{n+1}(S_g)) \xrightarrow{\text{cup product}} H^2(\text{PConf}_{n+1}(S_g))$.

The following lemma deals with $\text{Ker}(C_n)$.

Lemma 3.4. *Let $\{a_k, b_k\}$ be a symplectic basis for $H^1(S_g)$. For $1 \leq i, j \leq n$, we denote*

$$M_{i,j} = \sum_k p_i^* a_k \otimes p_j^* b_k - p_i^* b_k \otimes p_j^* a_k.$$

Then the kernel of C_n is spanned by the set $\{R_{i,j} := p_i^[S_g] + p_j^*[S_g] - M_{i,j} : 1 \leq i, j \leq n\}$. Furthermore, the set $\{R_{i,j}\} \subset \mathbb{Q}^n \oplus \bigoplus_{i \neq j} H_i^1 \otimes H_j^1$ are linearly independent as \mathbb{Q} vector space which gives us the following exact sequence:*

$$0 \rightarrow \bigoplus_{1 \leq i, j \leq n} \mathbb{Q}R_{i,j} \rightarrow \mathbb{Q}^n \oplus \bigoplus_{i \neq j} H_i^1 \otimes H_j^1 \xrightarrow{C_n} H^2(\text{PConf}_n(S_g))$$

Proof. The strategy of the proof is first to verify the relations and then to compute the rank of the image in order to check whether it satisfies the lemma.

Step 1 Verification of the relations.

We prove these relations when $n = 2$ and then use the projection $\text{PConf}_n(S_g) \rightarrow \text{PConf}_2(S_g)$ to get the relations for the n th case.

Let Δ be the diagonal in $S_g \times S_g$. Now

$$\text{PConf}_2(S_g) = S_g \times S_g - \Delta.$$

The long exact sequence for the cohomology of triple $(S_g \times S_g, S_g \times S_g - \Delta)$ is the following:

$$H^1(S_g \times S_g - \Delta) \rightarrow H^2(S_g \times S_g, S_g \times S_g - \Delta) \rightarrow H^2(S_g \times S_g) \rightarrow H^2(S_g \times S_g - \Delta)$$

By the Thom Isomorphism, $H^2(S_g \times S_g, S_g \times S_g - \Delta) = \mathbb{Q}$. Also the image of $H^2(S_g \times S_g, S_g \times S_g - \Delta) = \mathbb{Q}$ in $H^2(S_g \times S_g)$ is $p_1^*[S_g] + p_2^*[S_g] + M_{1,2}$ as explained in [MS74] Theorem 11.11. So we get the following isomorphism.

$$H^2(S_g \times S_g)/\mathbb{Q}R_{1,2} \xrightarrow{\text{injective}} H^2(S_g \times S_g - \Delta)$$

Therefore, the kernel of $H^2(S_g \times S_g) = \mathbb{Q}^2 \oplus H^1(S_g) \otimes H^1(S_g) \xrightarrow{C_2} H^2(\text{PConf}_2(S_g))$ is $p_1^*[S_g] + p_2^*[S_g] - M_{1,2}$.

Let $p_{i,j} : \text{PConf}_n(S_g) \rightarrow \text{PConf}_2(S_g)$ be the projection from n -tuples of ordered disjoint points on S_g to the i th and j th components. Since $\mathbb{Q}p_i^*[S_g] \oplus \mathbb{Q}p_j^*[S_g] \oplus H_i \otimes H_j$ is a pullback via this projection, C_n should have all the relations that C_2 has. Therefore we verify all the relations. If we have $\sum_{i \neq j} k_{i,j} R_{i,j} = 0 \in \mathbb{Q}^n \oplus \bigoplus_{i \neq j} H_i^1 \otimes H_j^1$, after projecting to $H_i \otimes H_j$, we would get $k_{i,j} M_{i,j} = 0$, which gives

The homomorphism $H^1(\text{PConf}_n(S_g)) \otimes H^1(S_g) \xrightarrow{\text{cup}} H^2(\text{PConf}_{n+1}(S_g))$ is the cup product of $\bigoplus_{i=1}^n H_i$ and H_{n+1} .

$\text{Im}(\text{cup})$ is transverse to $H^2(\text{PConf}_n(S_g))$ because the cup map is a partial section. Therefore

$$\begin{aligned} \text{Rank}(\text{Im}(C_{n+1})) &\geq \text{Rank}(\text{Im}(C_n)) + \text{Rank}(\text{Im}(\text{cup})) \\ &\geq \text{Rank}(\text{Im}(C_n)) + \dim(H^1(\text{PConf}_n(S_g)) \otimes H^1(S_g)) - \dim(H^1(S_{g,n})/H^1(S_g)) \\ &= \binom{n}{2}(2g \times 2g) + n - \binom{n}{2} + n(2g \times 2g) - (n-1) \\ &= \binom{n+1}{2}(2g \times 2g) + n + 1 - \binom{n+1}{2}. \end{aligned}$$

□

With this claim established, we complete our proof of the lemma. □

3.3 Step 3: Properties of the cup product

Definition 3.6. We call an element $x = x^1 + \dots + x^n \in \bigoplus_{i=1}^n H^1(S_g) = H^1(\text{PConf}_n(S_g))$ a *crossing element* if $\#\{i : x^i \neq 0\} > 1$, i.e. there exists H_i such that $x \in H_i$.

Lemma 3.7. Let $x = x^1 + \dots + x^n$ and $y = y^1 + \dots + y^n$ be two element in $H^1(\text{PConf}_n(S_g))$. Suppose that x or y is a crossing element. If $xy = 0 \in H^2(\text{PConf}_n(S_g))$, then x and y are proportional, i.e. $\lambda x = \mu y$ for some constants λ and μ .

Proof. We will prove this by induction.

$$xy = x^1y^1 + x^2y^2 + \dots + x^ny^n + \sum_{i \neq j} (x^iy^j - y^ix^j)$$

If we have $xy = 0$, we would have the following equality in $\mathbb{Q}^n \oplus \bigoplus_{i \neq j} H_i^1 \otimes H_j^1$.

$$x^1y^1 + x^2y^2 + \dots + x^ny^n + \sum_{i \neq j} (x^iy^j - y^ix^j) = \sum k_{i,j} R_{i,j} = \sum k_{i,j} (p_i[S_g] + p_j[S_g] + M_{i,j})$$

By the independence of all the terms in $\mathbb{Q}^n \oplus \bigoplus_{i \neq j} H_i^1 \otimes H_j^1$, we have

$$x^iy^j - y^ix^j = k_{i,j} M_{i,j}$$

If x^i and y^i are proportional, since $g > 1$, we have 4 terms in $M_{i,j}$, therefore we don't have enough basis to span our $M_{i,j}$. If x^i and y^i are independent, we can use a multiple of x^i and a multiple of y^i to make a symplectic basis. Since $g > 1$, we have 4 terms in $M_{i,j}$, therefore we don't have enough basis to span our $M_{i,j}$ either. Therefore $k_{i,j} = 0$ and $x^i \otimes y^j - y^i \otimes x^j = 0$ in $H_i^1 \otimes H_j^1$. Without loss of generality, we assume x a crossing element and $x_1 \neq 0$ and $x_2 \neq 0$. We break our proof into the following cases.

Case 1) $y^1 \neq 0$ and y^1 is not proportional to x^1

$x^1 \otimes y^j = y^1 \otimes x^j$ gives us that $y^j = 0$ and $x^j = 0$ for any j . However $x_2 \neq 0$. Therefore this case is invalid.

Case 2) $y^1 \neq 0$ and $\lambda x^1 = \mu y^1$

$x^1 \otimes y^j = y^1 \otimes x^j$ gives us that $\lambda x^j = \mu y^j$, which verifies our lemma.

Case 3) $y^1 = 0$

$x^1 \otimes y^j = y^1 \otimes x^j$ gives us that $y^j = 0$ for any j . This means $y = 0$ therefore x and y are also proportional. □

3.4 Step 4: Finishing the proof of Theorem 1.5

Lemma 3.8. *Let F_h be the free group with h generators and let S_r be the genus r surface. If we have a surjective $PB_n(S_g) \xrightarrow{S} \Gamma$ when $\Gamma = F_h$ with $h > 1$ or $\Gamma = \pi_1(S_r)$ with $r > 1$, and we also have $p_i^*(H^1(S_g)) \cap S^*(H^1(\Gamma)) \neq 0$, then S factors through some p_i .*

Proof. The proof of this lemma uses the same idea as [Joh99]. The method can also be found in the paper of [Sal15] at Lemma 3.3 and 3.4.

If there is a common nonzero cohomology element, we will have the following commutative diagram.

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{S} & F_h \\ \downarrow p_i & & \downarrow x \\ \pi_1(S_g) & \xrightarrow{y} & \mathbb{Q} \end{array}$$

Let K be the kernel K of p_i , which is a finitely generated normal subgroup of $PB_n(S_g)$. The image of K under S is also a normal subgroup of $\pi_1(F_h)$. However every finitely generated normal subgroup of F_h has finite index or is trivial. For the genus $r > 1$ surface group case, any nontrivial finitely-generated normal subgroup of $\pi_1(S_r)$ also has finite index (see Property (D6) in [Joh99]). If it has finite index, then after composing with x , the image of K won't be trivial; however K is the kernel of p_i so it has to go to zero.

If the image is trivial, we get a surjection $\pi_1(S_g) \rightarrow F_h$, therefore S factors through p_i . \square

To prove Theorem 1.5, we have to include a lemma talking about the possible image of the homomorphism.

Lemma 3.9. *Every finitely generated subgroup of $\pi_1(S_g)$ is either finitely generated free group F_h or surface group $\pi_1(S_r)$ with $r \geq g$. When $r = g$, the subgroup is the whole group $\pi_1(S_g)$.*

Proof. Subgroup of surface group is the same as cover of surface, so the subgroup is also going to be the fundamental group of some surface (possibly noncompact). In the noncompact case, it is free group. In the compact case, it is a finite cover. The Euler characteristic is multiplicative under cover, therefore $\chi(S_r) = n\chi(S_g)$. Since $g > 1$, if $n > 1$, we have $r > g$. If $n = 1$, this is trivial cover. \square

Proof of Theorem 1.5. Given a homomorphism $PB_n(S_g) \xrightarrow{S} \pi_1(S_g)$, according to Lemma 3.9, if $\text{Im}(S)$ is not \mathbb{Z} , the image has to be F_h with $h > 1$ or $\pi_1(S_r)$ with $r \geq g$. Furthermore, if S does not factor through any p_i , then by Lemma 3.8, $S^*(H^1(\text{Im}(S)))$ does not intersect nontrivially with any of the H_i^1 . This means that all nonzero elements of $S^*(H^1(\text{Im}(S)))$ are crossing elements. However $r \geq g > 1$ and $h > 1$ mean there are two crossing elements x and y in $S^*(H^1(\text{Im}(S)))$ that are independent and whose cup product is zero. Lemma 3.7 tells us this is impossible, which successfully proves 2) of Theorem 1.5.

Now to prove 1), we have a surjection homomorphism $PB_n(S_g) \xrightarrow{p_i} \pi_1(S_g) \xrightarrow{A} \pi_1(S_g)$ (since this homomorphism factors through some p_i). However surface groups are Hopfian which means that all surjective self homomorphisms between the surface group $\pi_1(S_g)$ must be an automorphism. Therefore A is an automorphism giving us 1) in [Theorem 1.5]. \square

4 The proof of Theorem 2.2

Since we already established all possible homomorphisms R in Theorem 1.5, the key idea of extending the homomorphism to $\text{PMod}_{g,n}$ is that it has to be equivariant with the action of Mod_g . We then use homology to rule out other possibilities.

If we can extend R , then for any $e \in \text{PMod}_{g,n}$ and $f \in PB_n(S_g)$ we have $R(efe^{-1}) = p(e)R(f)p(e)^{-1}$. The action of $\text{PMod}_{g,n}$ and $\text{Mod}_{g,n}$ on $PB_n(S_g)$ and $PB_n(S_g)$ are given by conjugation in the exact sequence (5) and (6), respectively. Therefore we have a commutative diagram:

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{e} & PB_n(S_g) \\ \downarrow R & & \downarrow R \\ \pi_1(S_g) & \xrightarrow{p(e)} & \pi_1(S_g) \end{array}$$

Since both e and $p(e)$ are isomorphisms of groups, $\text{Im}(R) = \text{Im}(R \circ e)$. This gives us the following diagram:

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{e} & PB_n(S_g) \\ \downarrow R & & \downarrow R \\ \text{Im}(R) & \xrightarrow{p(e)} & \text{Im}(R) \end{array} \tag{13}$$

Because of Lemma 3.9, we know that we have 4 possibilities for $\text{Im}(R)$: F_h for $h = 0, h > 0$ and $\pi_1(S_r)$ for $r = g$ or $r > g$. Now, let's do a case to case study.

Case 1) $\text{Im}(R) = 1$

Proof. In this case, we know that the whole subgroup $PB_n(S_g)$ goes to 1, so we have a homomorphism $\text{PMod}_{g,n}/PB_n(S_g) = \text{Mod}_g \rightarrow \text{Mod}_{g,1}$. However, the $n = 0$ case has already been proved, for example in [FM12] Corollary 5.11. \square

Case 2) $\text{Im}(R) = F_h$, while $h > 0$

Proof. Let a subspace $H \subset H^1(\text{PConf}_n(S_g); \mathbb{Z}) \cong \bigoplus_{i=1}^n H_i$ be an isotropic subspace if any $a, b \in H$, we have $a \cup b = 0 \in H^2(\text{PConf}_n(S_g))$.

Claim 4.1. Mod_g does not fix any isotropic subspace of $H^1(\text{PConf}_n(S_g))$.

Proof. If there exists $x \in H$ a crossing element, because of Lemma 3.7, we know that $xy = 0$ iff y is proportional to x . Therefore if H is isotropic, $H = \mathbb{Q}x \subset H^1(\text{PConf}_n(S_g); \mathbb{Z})$.

If $\dim(H) > 1$, H does not contain crossing elements. In this case, if there exist $x, y \in H$ and $i \neq j \in \{1, 2, \dots, n\}$ such that $x \neq 0 \in H_i$ and $y \neq 0 \in H_j$, we would have $x + y$ a crossing element. Therefore there exists i such that $H \subset H_i$.

Mod_g acts on $H^1(\text{PConf}_n(S_g); \mathbb{Z}) \cong \bigoplus_{i=1}^n H_i$ by acting on each component. We know that the action of Mod_g on $H^1(S_g)$ does not fix any isotropic subspace, therefore if $H \subset H_i$, Mod_g does not fix H . If $\dim(H) = 1$, Mod_g also does not fix it. \square

We have the following diagram.

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{e} & PB_n(S_g) \\ \downarrow R & & \downarrow R \\ F_h & \xrightarrow{p(e)} & F_h \end{array}$$

For every $e \in \text{PMod}_{g,n}$, the above diagram holds means that $R^*(H^1(F_h; \mathbb{Z})) \subset H^1(S_g; \mathbb{Z})$ has to be fixed under the action of Mod_g . This is impossible because $R^*(H^1(F_h)) \subset H^1(\text{PConf}_n(S_g))$ is an isotropic subspace of $H^1(\text{PConf}_n(S_g))$, but Mod_g does not fix any isotropic subspace. \square

Case 3) $\text{Im}(R) = \pi_1(S_g)$

Proof. If R is one of the forgetful homomorphism p_i , we have for any $e \in \text{PMod}_{g,n}$ and $f \in PB_n(S_g)$ that

$$p_i(e f e^{-1}) = p(e) p_i(f) p(e)^{-1}$$

We get that $p_i(e) p_i(f) p_i(e)^{-1} = p(e) p_i(f) p(e)^{-1}$. Therefore we find that $p(e)^{-1} p_i(e)$ commutes with $p_i(f)$ for any $f \in PB_n(S_g)$. The image of p_i on $PB_n(S_g)$ is the whole group $\pi_1(S_g)$. Therefore, $p(e)^{-1} p_i(e) \in \text{Mod}_{g,1}$ commutes with the subgroup $\pi_1(S_g)$. However, there is no element in $\text{Mod}_{g,1}$ that commutes with $\pi_1(S_g)$ except 1, so we get that $p(e)^{-1} p_i(e) = 1 \in \text{Mod}_{g,1}$. This tells us that $p = p_i$.

If R is one of the forgetful homomorphism p_i post-composing with an automorphism A , with a similar argument as above, we get that $p(e) = A p_i(e) A^{-1}$. Considering that the images of $A p_i(e) A^{-1}$ and $p_i(e)$ have to be equal in Mod_g for any e , we will have $A p_i(e) = p_i(e) A$ for any $e \in \text{Mod}_g$. Therefore, we will have $A \in \text{CenterMod}_g$. For $g > 2$, $\text{CenterMod}_g = 1$, therefore we will have $A \in \pi_1(S_g)$. For $g = 2$, we could have $A = \text{hyperelliptic element}$.

We will explain what a conjugacy means topologically in the last section. \square

Case 4) $\text{Im}(R) = \pi_1(S_r)$ while $r > g$

Proof. Because of Lemma 1.5, R factors through p_i . However there is no surjective homomorphism from $\pi_1(S_g) \rightarrow \pi_1(S_r)$ since $\text{Rank}(H^1(S_r)) > \text{Rank}(H^1(S_g))$. \square

5 The proof of Theorem 2.3

In this section, we begin with the proof of Theorem 2.3

Lemma 5.1.

$$H^1(\text{Conf}_n(S_g)) \cong H^1(S_g)$$

and the image of $H^1(\text{Conf}_n(S_g)) \rightarrow H^1(\text{PConf}_n(S_g))$ is equal to the image of the diagonal map, i.e. $x \rightarrow (x, x, \dots, x) \in H^1(\text{PConf}_n(S_g)) \cong \bigoplus_{i=1}^n H_i$.

Proof. Since $\text{Conf}_n(S_g) = \text{PConf}_n(S_g) / \Sigma_n$, we can use the transfer map to get

$$H^1(\text{Conf}_n(S_g)) = H^1(\text{PConf}_n(S_g))^{\Sigma_n}.$$

It is not hard to see that the Σ_n invariant subspace of $H^1(\text{PConf}_n(S_g))$ is the diagonal subspace. \square

Proof of Theorem 2.3. If we have a homomorphism p' in the diagram in Theorem 2.3, after composing an injection $\text{PMod}_{g,n} \xrightarrow{i} \text{Mod}_{g,n}$, we get a homomorphism p as in Theorem 2.2. We have already proved Theorem 2.2 in the last section that this composition has to be equal to a conjugate by $A \in \pi_1(S_g)$ of p_i . Let C_A denote the conjugate by A action on $\pi_1(S_g)$, we have $p' \circ i = C_A \circ p_i$. Restricting to the kernel of π_n and π'_n , the following diagram holds.

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{p_i} & B_n(S_g) \\ \downarrow i & & \downarrow R' \\ \pi_1(S_g) & \xrightarrow{C_A} & \pi_1(S_g) \end{array}$$

The image of $H^1(S_g) \xrightarrow{(A \circ p_i)^*} H^1(\text{PConf}_n(S_g))$ is H_i ; however the image of $H^1(\text{Conf}_n(S_g)) \rightarrow H^1(\text{PConf}_n(S_g))$ as described in the previous lemma is the diagonal. Thus this is a contradiction. \square

6 Going back to topology

Now that the group theoretical theorem has been proved, we are ready to translate everything back to topology and try to understand the conjugacy.

Proof of Theorem 1.1 and 1.3. The original problem is to find sections of the universal bundles (1) and (2). Since all the spaces involved are $K(\pi, 1)$ spaces, it is the same as finding all the splittings s , i.e. $u \circ s = id$ and s' , i.e. $u' \circ s' = id$ of the following exact sequences.

$$1 \rightarrow \pi_1(S_g) \rightarrow \text{CPMod}_{g,n} \xrightarrow{u} \text{PMod}_{g,n} \rightarrow 1.$$

$$1 \rightarrow \pi_1(S_g) \rightarrow \text{CMod}_{g,n} \xrightarrow{u'} \text{Mod}_{g,n} \rightarrow 1.$$

Furthermore, in Section 2, we show that this problem is equivalent to finding homomorphisms p and p' satisfying the diagram (7) and (8).

Theorem 2.3 shows that there is no homomorphism p' satisfying the diagram (8), therefore we get Theorem 1.3.

To prove Theorem 1.1, first note that Theorem 2.2 shows that every homomorphism p that satisfies the diagram (7) is a conjugate of some p_i by an element $A \in \pi_1(S_g)$. In the splitting situation, it is saying that any splitting s is a conjugate of s_i by $A \in \pi_1(S_g)$.

Claim 6.1. *Geometrically, conjugation by $A \in \pi_1(S_g)$ to an existing section s means point pushing along one fiber by loop A . Therefore, topologically this section is homotopic to s .*

Proof. Given a manifold M with a base point b , there is a special subgroup of the group of isotopy of M fixing b , the point pushing subgroup.

$$\pi_1(M, b) \xrightarrow{\text{point pushing}} \text{Isotopy}(M, b)$$

Every isotopy of M induces an automorphism on $\pi_1(M)$. The image of a point pushing in $\text{Aut}(\pi_1(M))$ is the conjugacy by the loop.

In our case, we mark a base point in the base and also in the total space of our universal bundle. We will call them b and b' .

The effect of conjugating by A of a section s is to point push our base point b' around A in the fiber. However, this section is going to be homotopic to our original section s because the point pushing is homotopic to identity if we forget about the base point.

□

This proves Theorem 1.1.

□

References

- [Che17] L Chen. Birman exact sequence doesn't split on finite index subgroup. *In Preparation*, 2017.
- [EE67] C. J. Earle and J. Eells. The diffeomorphism group of a compact Riemann surface. *Bull. Amer. Math. Soc.*, 73:557–559, 1967.
- [EK76] C. J. Earle and I. Kra. On sections of some holomorphic families of closed Riemann surfaces. *Acta Math.*, 137(1-2):49–79, 1976.
- [FM12] B Farb and D Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
- [Joh99] F. E. A. Johnson. A rigidity theorem for group extensions. *Arch. Math. (Basel)*, 73(2):81–89, 1999.
- [Mes90] G Mess. Unit tangent bundle subgroups of the mapping class group. 1990.
- [MS74] J Milnor and J Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76.
- [Sal15] N Salter. Cup products, the Johnson homomorphism and surface bundles over surfaces with multiple fiberings. *Algebr. Geom. Topol.*, 15(6):3613–3652, 2015.

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